Introduction to Mathematics and Modeling

lecture 6

Integrals

UNIVERSITY OF TWENTE.

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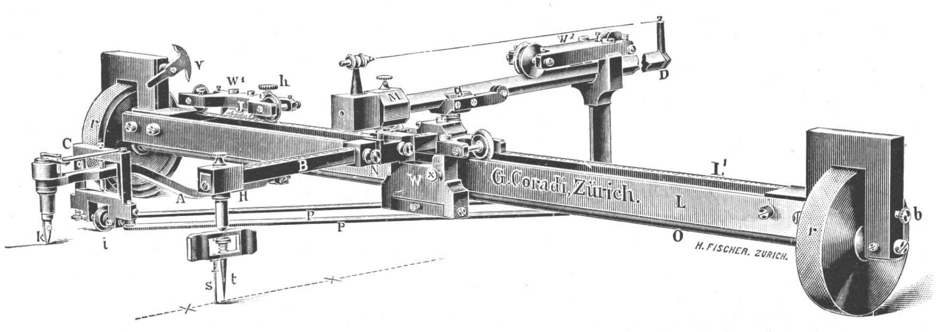
This week



The integraph, an instrument for measuring integrals

1. Section 5.1: area estimating with finite sums
2. Section 5.2: limits of finite sums
3. Section 5.3: the definite integral
4. Section 5.4: the fundamental theorem of calculus

The Σ-notation

We can write sums with the Σ-notation:

$$\sum_{k=M}^{N} a_k = a_M + a_{M+1} + a_{M+2} + \cdots + a_{N-1} + a_N$$

- Σ is the Greek letter "S" (pronounced as 'sigma'), which refers to "Sum".
- k is called the **index**.
- The index starts counting at M and stops counting at N.
- a_k is the k**-th term** of the sum, and is a formula containing k.
- If $N < M$ then the sum is equal to 0 by definition.
- The index is a *dummy*:

$$\sum_{k=3}^{6} a_k = \sum_{p=3}^{6} a_p = a_3 + a_4 + a_5 + a_6$$

The Σ-notation

$$\sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$$

$$= 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144$$

$$= 650.$$

Examples

- $\sum_{k=1}^{4}(-1)^k k =$

- $\sum_{k=1}^{2}\frac{k}{k+1} =$

Arithmetic series

Theorem

The sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$.

- Define S_n as the sum of the first n positive integers:

$$S_n = 1 + 2 + \cdots + (n-1) + n.$$

- with Σ-notation:

$$S_n = \sum_{k=1}^{n} k.$$

- Write out the terms in S_n *twice*:

$$\begin{aligned} S_n &= 1 + 2 + \cdots + (n-1) + n \\ S_n &= n + (n-1) + \cdots + 2 + 1 \end{aligned}$$

Rules 1.5

- Sum- and difference rule:

$$\sum_{k=M}^{N}(a_k+b_k)=\sum_{k=M}^{N}a_k+\sum_{k=M}^{N}b_k,\ \text{and}\ \sum_{k=M}^{N}(a_k-b_k)=\sum_{k=M}^{N}a_k-\sum_{k=M}^{N}b_k.$$

- Constant multiple rule:

$$\sum_{k=M}^{N} c\,a_k = c\sum_{k=M}^{N} a_k.$$

- Constant value rule:

$$\sum_{k=M}^{N} c = (N-M+1)c.$$

- Splitting rule:

$$\sum_{k=M}^{N} a_k = \sum_{k=M}^{P} a_k + \sum_{k=P+1}^{N} a_k.$$

Example 1.6

- Define T_n as the sum of the first n odd integers:

$$T_n = 1+3+\cdots+(2n-1).$$

- Notice that

$$\begin{aligned} T_n+(2+4+\cdots+2n) &= 1+2+3+\cdots+(2n-1)+2n \\ &= \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2+n \end{aligned}$$

- Furthermore

$$\begin{aligned} 2+4+\cdots+2n &= \sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k \\ &= 2\cdot\frac{n(n+1)}{2} = n(n+1) = n^2+n. \end{aligned}$$

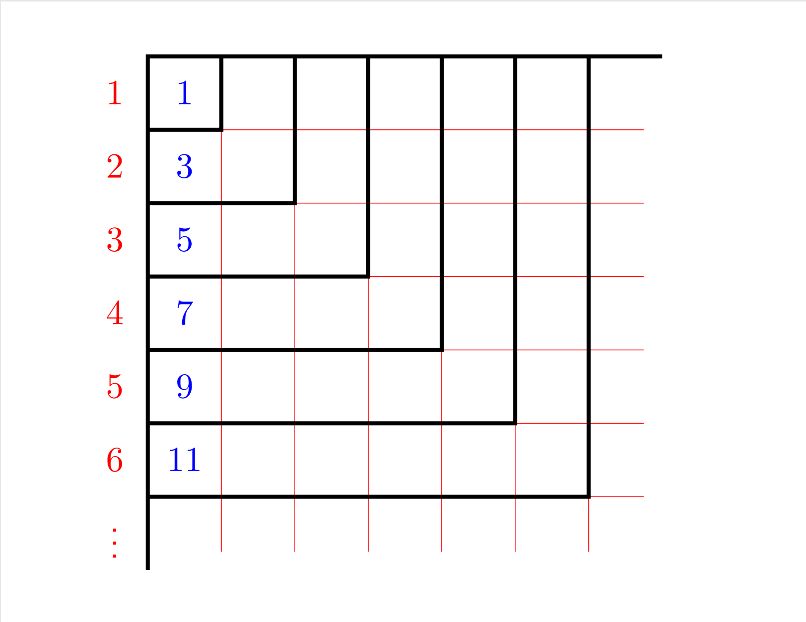
- Therefore

$$T_n = (2n^2+\cancel{n})-(n^2+\cancel{n}) = n^2.$$

Example

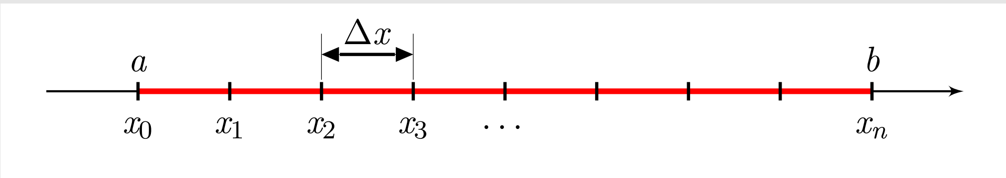
1.7

The sum of the first n odd integers is equal to n^2:



Partitions

2.1



Definition

A **partition of the interval** $[a, b]$ **in** n **subintervals** is a sequence $x_0, x_1, \ldots, x_n$ constructed as follows:

(i) $\Delta x = \dfrac{b-a}{n}$

(ii) $x_k = a + k\Delta x \qquad (k = 0, 1, \ldots, n)$

- Note that
$$x_0 = a, \qquad x_n = b, \qquad x_k - x_{k-1} = \Delta x.$$
- The number Δx is called the **mesh** of the partition.

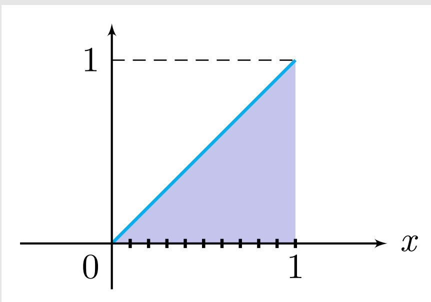
Riemann sums

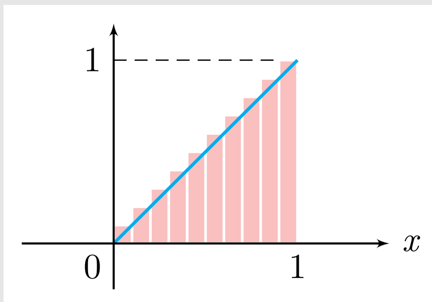
2.2

Integrals as limits of Riemann sums

3.1

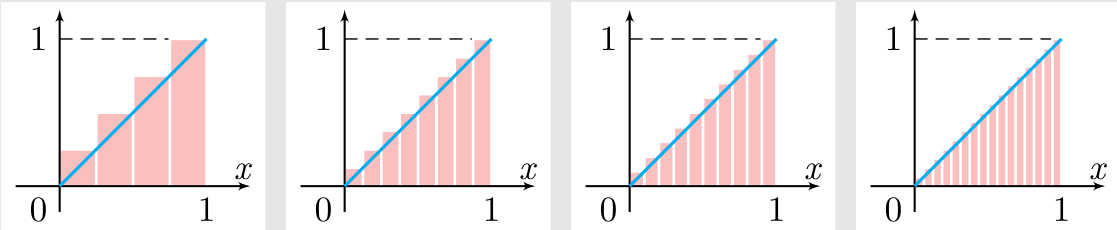
Approximate the area of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$ with a Riemann sum.





- Define the partition $x_k = k\Delta x = \dfrac{k}{n}$ with $\Delta x = \dfrac{1}{n}$.
- The Riemann sum of $f(x) = x$ is

$$\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}.$$

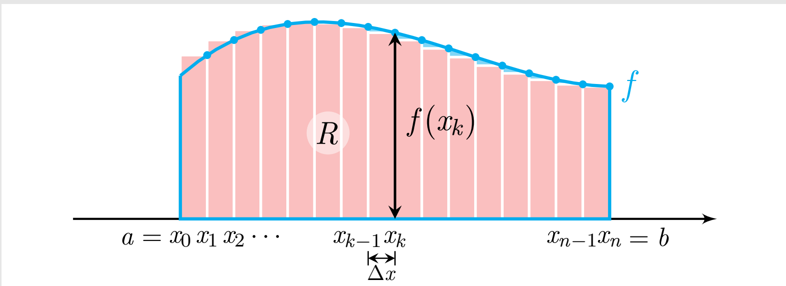


- Evaluate the Riemann sum:

$$\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} =$$

- If we let n approach infinity then

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k \Delta x =$$



- For a positive function, a Riemann sum can be regarded as the approximation of the surface area of the region R bounded by the graph of f, the x axis, and the lines $x = a$ and $x = b$.

Definition

The **definite integral of f over the interval** $[a, b]$ *is defined as*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(x_k) \cdot \Delta x \right)$$

- A definite integral can be regarded as the area of the region R.

Laws of integration

- The variable in the integral is a *dummy*:

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^b f(u)\ \mathrm{d}u$$

- Linearity:

$$\int_a^b \alpha f(x) + \beta\, g(x)\ \mathrm{d}x = \alpha \int_a^b f(x)\ \mathrm{d}x + \beta \int_a^b g(x)\ \mathrm{d}x$$

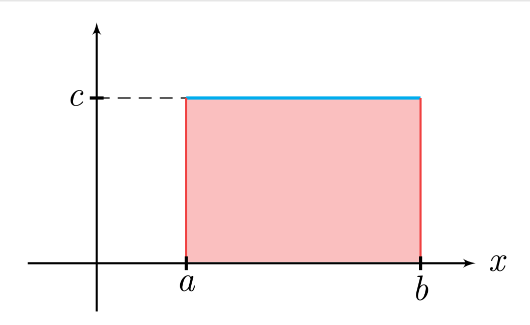
- Additivity:

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x$$

- Interchanging the upper and lower limit gives a minus sign:

$$\int_a^b f(x)\ \mathrm{d}x = -\int_b^a f(x)\ \mathrm{d}x$$

Constant functions

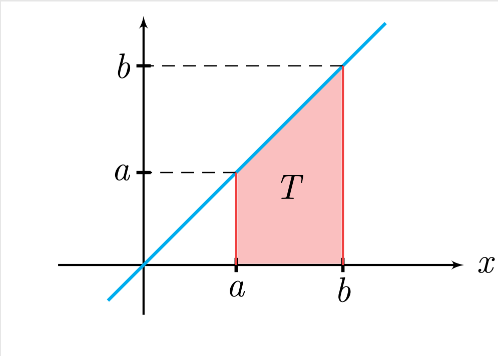


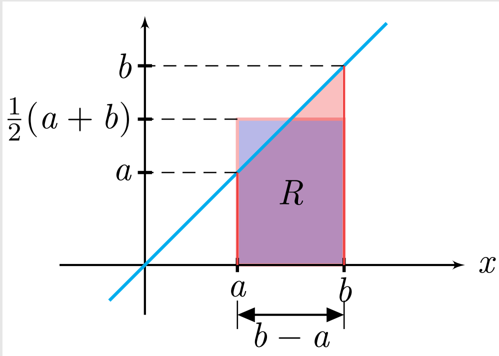
$$\int_a^b c\ \mathrm{d}x = c(b-a)$$

Notice that the Riemann sum of *any* partition is

$$\sum_{k=1}^{n} c\,\Delta x = n \cdot c\Delta x = c\,n\frac{b-a}{n} = c(b-a).$$

Laws of integration

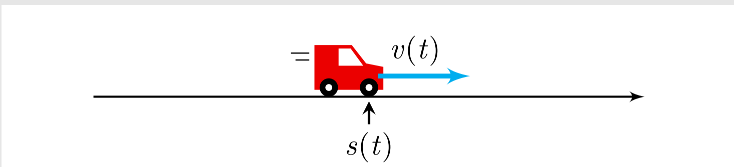




$$\int_a^b x \, \mathrm{d}x = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2$$

- Note that $\frac{1}{2}b^2 - \frac{1}{2}a^2 = \frac{1}{2}(b+a)(b-a) = \text{Area}(R)$.
- The regions T and R have the same area.

Displacement and velocity



- Differentiate displacement to compute velocity:

$$v(t) = s'(t)$$

- The displacement can be computed from the velocity by integrating:

$$s(t) = \lim_{n\to\infty} \sum_{k=1}^{n} v(t_k)\Delta t = \int_0^t v(\tau) \, \mathrm{d}\tau$$

The integeral $\int_0^t v(\tau) \, \mathrm{d}\tau$ is a function $s(t)$ whose derivative is v.

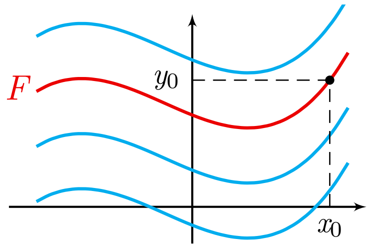
Definition

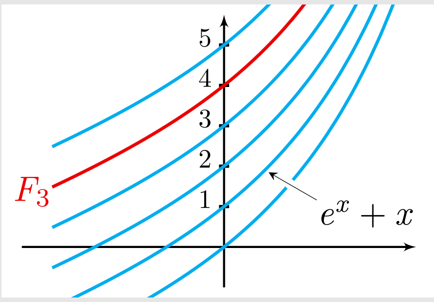
We call a function F an ***antiderivative*** *for f if $F'(x) = f(x)$.*

- Antiderivatives are not unique. If F is an antiderivative for f, then so is $F(x) + C$ for any constant C:
$$\frac{d}{dx}(F(x) + C) = F'(x) = f(x).$$

Theorem

Let (x_0, y_0) be a point in the plane. Then there is a unique antiderivative F of f for which $F(x_0) = y_0$.





- Let $f(x) = e^x + 1$, then $F(x) = e^x + x$ is an antiderivative of f.
- For arbitrary C the function
$$F_c(x) = e^x + x + C$$
is also an antiderivative of f.
- There is only one antiderivative of f for which $F(0) = 4$:
$$F(x) = e^x + x + 3.$$
- The correct value for C is found by solving the equation $F_c(0) = 4$:
$$4 = F_c(0) = e^0 + 0 + C = 1 + C,$$
hence $C = 3$.

The Fundamental Theorem of Calculus

1. *Define the function*

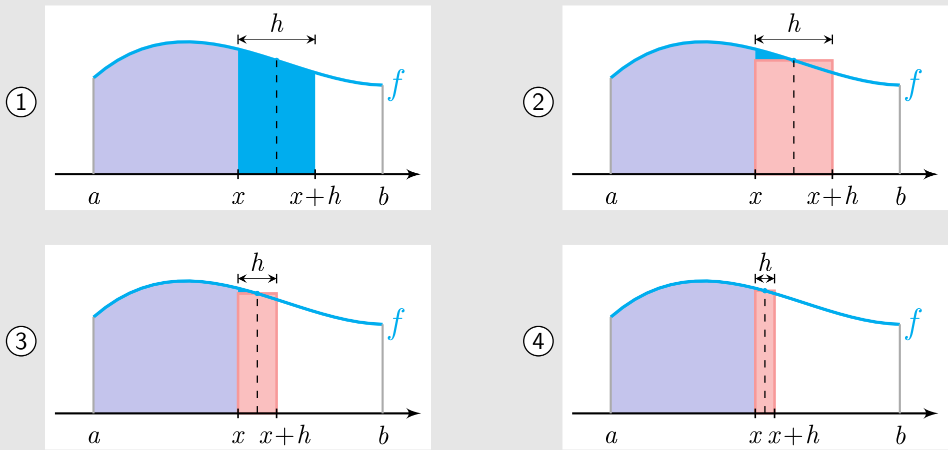
$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

then F is an antiderivative for f, in other words: $F'(x) = f(x)$.

2. *If F is an antiderivative for f then*

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

- Notation: $F(b) - F(a) = \Big[\, F(x) \,\Big]_a^b = F(x)\,\Big|_a^b$.
- The function $F(x) = \int_a^x f(t)\,\mathrm{d}t$ also satisfies $F(a) = 0$, so F is the *unique* antiderivative of f for which $F(a) = 0$.



- Define $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t = f(x).$$

Integrals with e^x 4.6

The fundamental theorem of Calculus:

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a) \quad \text{where } F' = f.$$

- $\int_0^{\ln 2} e^x \,\mathrm{d}x =$

Integrals with $\sin$ and $\cos$ 4.7

The fundamental theorem of Calculus:

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a) \quad \text{where } F' = f.$$

- $\int_0^{\pi/2} \cos(x)\,\mathrm{d}x =$

- $\int_\pi^{2\pi} \sin(x)\,\mathrm{d}x =$

Power functions

4.8

- Notice that for arbitrary real α we have

$$\frac{d}{dx}\left(x^{\alpha+1}\right) = (\alpha+1)x^{\alpha}.$$

- Hence, if $\alpha \neq -1$:

$$\frac{d}{dx}\left(\tfrac{1}{\alpha+1}x^{\alpha+1}\right) = x^{\alpha}.$$

- The antiderivative of x^{α} is: $\dfrac{1}{\alpha+1}x^{\alpha+1} + C$ if $\alpha \neq -1$.

- The antiderivative of $x^{-1} = \dfrac{1}{x}$ is: $\ln|x| + C$.

See lecture 5

Integrals with powers of x

4.9

The fundamental theorem of Calculus:

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a) \quad \text{where } F' = f.$$

- $\int_0^2 x^3\,\mathrm{d}x =$

Polynomials

- $\int_0^1 2x^3 - 2x + 1 \; \mathrm{d}x =$